Extending the Non-Linear Beam Dynamics Concept of 1D Fix-Points to 2D Fix-Lines

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The origin of non-linear dynamics traces back to the study of the dynamics of planets with the seminal work of Poincaré at the end of the 19th century: "Les Méthodes Nouvelles de la Mécanique Céleste", Vols. 1-3 (Gauthier Villars, Paris) (1899). In his work he introduced a methodology fruitful to investigate the dynamical properties of complex systems which led to the so called "Poincaré surface of section", which allows to capture the global dynamical properties of a system, characterized by fix-points and separatrix as far as it concerns regular and chaotic motion. For two dimensional phase space (1 degree of freedom) this approach has been extremely useful and applied to particle accelerators for controlling their beam dynamics as of the second half of the 20th century. We describe here an extension of the concept of 1D fix-points to fix-lines in 2D. These structures become the fundamental entities for characterizing the non-linear motion in the four dimensional phase space (2 degrees of freedom).

For experimental physicists, accelerators are devices to provide particle beams to a detector, or to bring two beams to collision to study fundamental properties of matter at subatomic level. It is less known that particles in a circular accelerator are subject to violent and complex dynamics, which in many aspects resemble and even exceed the complexity of the dynamics of planet motion around the sun. In accelerators, magnet non-linearities are responsible for the non-linear dynamics in 1, 2 and even 3 degrees of freedom. The characterization of the nonlinear motion is usually expressed in terms of the stability properties and trajectory deformation. Accelerator physicists have become familiar with properties such as dynamic aperture, fix-points, island and separatrix structures, as the requirement of controlling the beam dynamics. This has led to the development of the non-linear dynamics "for accelerators" [1–4]. Particle motion can either be regular or chaotic, the latter leading to particle loss. The characterization of the regular motion and the onset of chaotic motion are therefore of utmost importance for particle accelerators [1, 2].

In absence of non-linearities the horizontal linear motion is governed by a harmonic oscillator equation of motion $x'' + k_x(s)x = 0$, where $k_x(s)$ is the strength of the linear focusing/defocusing elements in the accelerator, resulting in the linear horizontal tune (phase advance per turn), $Q_x = 1/(2\pi) \int_0^L \beta_x(s)^{-1} ds$, with $\beta_x(s)$ the well known horizontal beta function incorporating the focusing system $k_x(s)$, and L the length of the circular accelerator [5]. The particle trajectories in the 2D phase space are lines around a closed-orbit which typically is located at zero amplitude. In the Poincaré surface of section [6] (any fixed longitudinal location s) one finds ellipses around the closed-orbit.

The introduction of non-linearities disturb this simple and elegant structure. The motion of a particle is governed by the non-linear equation $x'' + k_x(s)x = k_n(s)x^n$, with $k_n(s)$ the strength of the non-linearity of order n.

The topology of the orbits in the Poincaré surface of section is determined by the tune Q_x, which is now defined as the averaged phase advance per turn. For Q_x close to the resonance $nQ_x = m$ the topology of the orbits changes and some of the Courant-Snyder ellipses are broken into higher order closed-orbits at non-zero amplitudes (fix-points) and the particle motion in its vicinity is restricted to islands around the fix-points. Lastly, at some distance to these fix-points a separatrix is reached that separates the island motion from the nominal motion around the closed-orbit at zero amplitude. It is interesting to note that chaotic motion develops around the separatrix due unavoidable additional resonances as originally described by Poincaré in 1899 [6].

The Poincaré surface of section method allows a visual representation of the dynamics of a particle, which is useful for describing the motion. The control of the islands and separatrix is even used for operational schemes like the slow resonance extraction [7].

The importance of the nonlinear dynamics in accelerators has recently been emphasized with the advent of new accelerator projects [8, 9], which have turned the focus on machines, where properties vary with time. The phenomenon of 1D resonance crossing for high intensity beams has been explained in terms of trapping and scattering of a particle interacting with islands, and extensive experimental and numerical studies have characterized the case of 1D dynamics [10, 11].

The reduction of the dynamics to one degree of freedom allows a visual representation of the main feature of the dynamics in terms of islands and fix-points. However, the dynamics in accelerators is actually determined in 2 and 3 degrees of freedom and the resonances excited by non-linear components in an accelerator are found for the horizontal and vertical tunes Q_x, Q_y satisfying the relation $n_x Q_x + n_y Q_y = N$. A theory describing the dynamics of a particle in the 2D has been developed in the 1950s by Schoch [4]. This theory describes how the strength of

magnet non-linearities is related to the resonant behavior of a particle: for non-linear field components distributed around the circular accelerator, the main properties of a resonance can be related to a quantity called "driving term". The theory predicts the resonance stop-band as a function of the driving term. Despite this enormous progress it remains un-explained how resonance structures in two or more degrees of freedom actually govern the phase space. The purpose of this letter is to shed some new light onto this issue.

The discussion of the dynamics in terms of islands and fix-points is now more difficult, the Poincaré surface of section is a 4D phase space and no longer easily understandable. One can try to predict the beam evolution via computer simulations to circumvent the complicate theoretical analysis. In fact, the actual development of the treatment of the non-linear dynamics in accelerators is quite mature [12], and the present level of computer power allows demanding numerical investigation. However, the increase of computer power does not help in understanding what really happens in the multi-dimensional phase space if we do not know what to look for in the complex dynamics unless guided by a deeper theoretical understanding of these complexities.

In the following an approach is presented on how to characterize the dynamics in the proximity of a coupled 2D resonance in a rigorous analytic way. The technical details of this approach will be discussed in an upcoming article [13].

The simplest situation of coupled non-linear dynamics is found when considering a circular accelerator affected by a sequence of thin sextupoles. In this case the equations of motion read

$$
\begin{aligned}
\frac{d^2x}{ds^2} + k_x(s)x &= -\frac{1}{2}k_2(s)(x^2 - y^2), \\
\frac{d^2y}{ds^2} + k_y(s)y &= k_2(s)xy.
\end{aligned}
\tag{1}
$$

The solution of this equation can be written as

$$
\begin{aligned}
x(s) &= \sqrt{\beta_x a_x}\cos[\phi_x(s) + \varphi_x], \\
y(s) &= \sqrt{\beta_y a_y}\cos[\phi_y(s) + \varphi_y],
\end{aligned}
\tag{2}
$$

where β_x, β_y are the well known horizontal and vertical beta functions, and the phase advances are $\phi_x(s) = \int_0^s \beta_x(s)^{-1}ds$, $\phi_y(s) = \int_0^s \beta_y(s)^{-1}ds$. The function $k_2(s)$ describes the local strength of the non-linear errors, which excite the resonance $Q_x + 2Q_y = N$, with N the harmonics of the distribution of the errors.

The quantities $a_x, a_y, \varphi_x, \varphi_y$ are the constants of linear motion. However, if we consider the system described by Eq.(1), and search for solutions of the form of Eq.(2), then $a_x, a_y, \varphi_x, \varphi_y$ are forced to become time dependent. The evolution of these "new" variables is governed by an Hamiltonian composed by a large number of harmonics, each of which has frequencies obtained as a combination of the tunes Q_x, Q_y. This is the hamiltonian of the nonlinear terms in Eq. (1). The situation is hopelessly complex for an analytic treatment when all these harmonics should be included into the equations of motion. However, for a machine set close to the resonance $Q_x + 2Q_y = N$, the harmonics with frequency $Q_x + 2Q_y - N$ become very slow. As a result only two slow varying harmonics remain while the others "average out" fast and are ignored. This approximation is valid close to the resonance, and for reasonable small non-linear errors. We then take the truncated Hamiltonian (slowly varying) as an approximation of the real Hamiltonian when the system is in proximity to a third order 2D coupled resonance. The same procedure is used in Refs. [4, 14]. In our study we find that N_j sextupolar errors lead to the slowly varying Hamiltonian:

$$
H_{s1} = \Lambda\sqrt{a_x}a_y\cos\left(\alpha + 2\pi\Delta_r\frac{s}{L} + \varphi_x + 2\varphi_y\right),
\tag{3}
$$

with $\Delta_r = Q_x + 2Q_y - N$ defined as the distance of the tunes Q_x, Q_y to the resonance. Here Λ is the driving term of the resonance, and α is its orientation in the complex plane. Both these factors incorporate the effect of the distributed sextupolar errors of integrated strength K_{2j} located at the position s_j. The quantity Λ, and the angle α determine the dynamics in this approximation.

The dynamics of the variables $a_x, a_y, \varphi_x, \varphi_y$, is given by the canonical equations of the Hamiltonian $2H_{s1}$, where H_{s1} is the slowly varying Hamiltonian Eq.(3), which is time dependent, and a procedure to remove the time dependency is desirable in order to identify invariants of motion.

Differently from previous works, we find that there is an infinite set of canonical transformations of the from

$$
\begin{aligned}
\tilde{a}_x &= a_x, \quad \tilde{\varphi}_x = \varphi_x + t_x 2\pi\Delta_r s/L, \\
\tilde{a}_y &= a_y, \quad \tilde{\varphi}_y = \varphi_y + t_y 2\pi\Delta_r s/L,
\end{aligned}
\tag{4}
$$

all suitable for creating a system of canonical coordinates $\tilde{a}_x, \tilde{\varphi}_x, \tilde{a}_y, \tilde{\varphi}_y$ where the Hamiltonian Eq.(3) becomes time independent. The coefficients t_x, t_y shall here satisfy the condition $t_x + 2t_y = 1$. Therefore, by assigning a value to $t_x = \tau$ we completely fix one possible system of canonical coordinates.

In the system τ, the new coordinates are subjected to the dynamics of a time-independent Hamiltonian via the canonical equations. The simplest solution of the canonical equations $\tilde{a}'_x = \tilde{\varphi}'_x = \tilde{a}'_y = \tilde{\varphi}'_y = 0$ is found to be

$$
\begin{aligned}
\tilde{a}_y &= \frac{(2\pi\Delta_r)^2}{4\Lambda^2 L^2}\tau(1 - \tau), \\
\tilde{a}_x &= \frac{(2\pi\Delta_r)^2}{16\Lambda^2 L^2}(1 - \tau)^2, \\
\tilde{\varphi}_x + 2\tilde{\varphi}_y + \alpha &= \pi M,
\end{aligned}
\tag{5}
$$

with M an integer of either 0 or 1. The values of τ are limited to $0 \leq \tau \leq 1$.

Any one of these solutions $\tilde{a}_x, \tilde{\varphi}_x, \tilde{a}_y, \tilde{\varphi}_y$ can be transformed to the laboratory frame via Eqs.(4, 2) at a specific longitudinal position along the accelerator. To simplify the discussion, but without losing generality, we take the

$x - y$ projection of the Poincaré surface of section and find that the particle coordinates lie on the curve

$$x(t) = \sqrt{\beta_x a_x} \cos(-2t + \pi M),$$
$$y(t) = \sqrt{\beta_y a_y} \cos(t), \tag{6}$$

where t is a variable that parameterizes the curve. We find that after each accelerator turn an initial point $(x, x', y, y')_0$, identified by $\tilde{a}_x, \tilde{a}_y, \tilde{\varphi}_x, \tilde{\varphi}_y$ and satisfying Eq.(5), is mapped to another point $(x, x', y, y')_1$ and all such points belong to a closed one dimensional curve, which in the $x - y$ projection has the analytic form of Eq.(6). At the same time $(x, x', y, y')_1$ in the system τ keep unchanged the value of the variables $\tilde{a}_x, \tilde{a}_y, \tilde{\varphi}_x, \tilde{\varphi}_y$. Similar result can be found in any projection of Poincaré surface of section.

We call this curve a 2D "fix-line" [15] based on its property that the particle coordinates will remain on this curve indefinitely. In analogy to 1D fix-points, we find that 2D fix-lines can be either stable or unstable. Fig.(1) shows an example of several projections of a Poincaré surface of section of a stable 2D fix-line. For convenience we show the classical horizontal and vertical projections $x - x'$, $y - y'$ in part a), and b), which describe circles guaranteeing that $\tilde{a}_x, \tilde{a}_y$ are constant. Part c) shows the $x - y$ projection, part d) shows the $x' - y'$ projection.
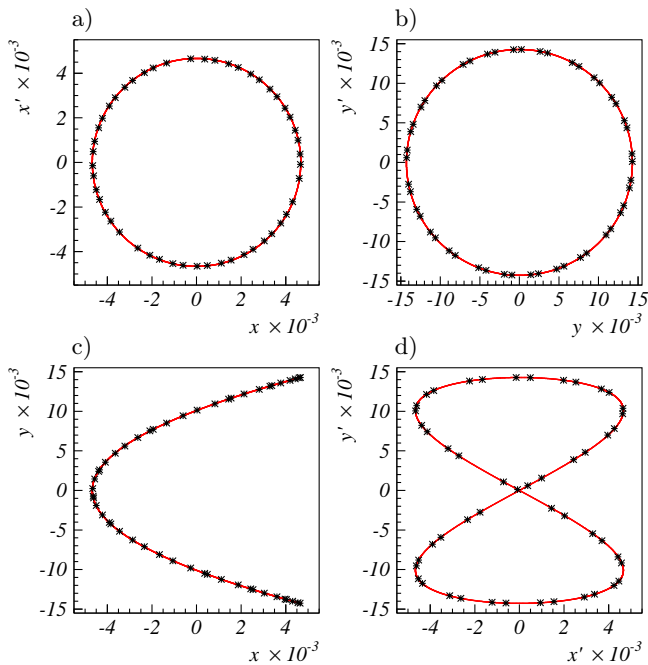

Figure 1: The 2D fix-line as predicted by the theory (red dots) and position of a particle from simulations (black markers). The circular shape in Part a), b) shows that $\tilde{a}_x, \tilde{a}_y$ are constants of the motion.

We have therefore reached the situation where in each system τ there is one stationary point that in the laboratory frame becomes a distinct fix-line. Therefore we can conclude that the 4D phase space is populated by an infinite set of fix-lines dependent on the parameter τ. These lines are all close to one another so as to form a complex structure in the 4D phase space.

It is an interesting feature of the canonical equations in the system τ that they lead to the invariant of motion $2\tilde{a}_x - \tilde{a}_y = C$. If we consider the same particle in another system τ', the value of $C, \tilde{a}_x$, and $\tilde{a}_y$ remain the same. This is due to the fact that the parameter τ does not play a physical role, while the variables $\tilde{a}_x, \tilde{a}_y$ have a physical meaning of particle emittance (or action). This is not the case for the variables $\tilde{\varphi}_x, \tilde{\varphi}_y$, which depend on τ. As the physics should not be dependent on τ, we combine $\tilde{\varphi}_x, \tilde{\varphi}_y$ to eliminate t_x, t_y by taking $\Omega = \tilde{\varphi}_x + 2\tilde{\varphi}_y$: Ω has the same value in any system τ (see Eq.(4)). This means that the variables $\tilde{a}_x, \Omega$ are more convenient to characterize the dynamics as they are independent of τ. Re-expressing the time independent Hamiltonian of any system τ into these variables we find a function neither dependent on time nor on τ, i.e. an invariant of motion $I(a_x, \Omega)$. In the laboratory frame we can re-cast it in conveniently "scaled" coordinates as:

$$\hat{I}(\hat{a}_x, \Omega) = \mu \sqrt{\hat{a}_x}(\hat{a}_x - \xi) \cos(\Omega + \alpha) + \hat{a}_x, \tag{7}$$

with $\hat{a}_x = [4L\Lambda/(2\pi\Delta_r)]^2 a_x$, $\hat{a}_y = [4L\Lambda/(2\pi\Delta_r)]^2 a_y$, $\xi = [4L\Lambda/(2\pi\Delta_r)]^2 C/2$, and $\mu = \Delta_r/|\Delta_r|$. Given the initial coordinates of a particle, we fix $\hat{I}, \xi$, hence the associated level line of Eq.(7) yields the particle trajectory in $\hat{a}_x, \Omega$ coordinates. This allows to predict the stability of all particles according to the value of the two invariants $\hat{I}, \xi$. An analysis of the stability of motion using Eq.(7) reveals that the edge of stability of particles is consistent with the position of the fix-lines. This is shown in part a) of Fig.(2) where the set of the fix-lines described by Eq.(5) forms a parabolic-like curve (color blue, and pink) in the $(\hat{a}_x, \hat{a}_y)$ plane. If a particle has initial values $\hat{a}_x, \hat{a}_y$ "inside" the set of the fix-lines, its stability depends on the value of Ω. The pink line is the collection of $\hat{a}_x, \hat{a}_y$ with unstable fix-lines, and the blue line is the collection of the values $\hat{a}_x, \hat{a}_y$ with stable fix-lines. The color code in the pictures provides the set of allowed Ω as function of $\hat{a}_x, \hat{a}_y$. This range is determined by the level line associated with the invariant $\hat{I}$ of a fix-line according to the following procedure: given a particle with certain $\hat{a}_x, \hat{a}_y$ values we find $\xi = \hat{a}_x - \hat{a}_y/2$, which identify a line that always intercept the pink line collection of the unstable fix-lines. To that interception point corresponds one value of τ, and one value $\hat{I}_{fl}$ of the invariant Eq.(7). The angles Ω for which the motion of a particle with initial a_x, a_y is stable are found in an interval of size $2\Delta\Omega$, with

$$\Delta\Omega(\hat{a}_x, \hat{a}_y) = \arccos\left[\frac{1}{\mu} \frac{\hat{I}_{fl} - \hat{a}_x}{\sqrt{\hat{a}_x}(\hat{a}_x - \xi)}\right], \tag{8}$$

where we take $0 \leq \Delta\Omega \leq \pi$. Part a) of Fig.(2) shows a red wide region where $\Delta\Omega = \pi$, there Ω can assume any values and the particle remain stable; other regions of the picture exhibit partial stability.
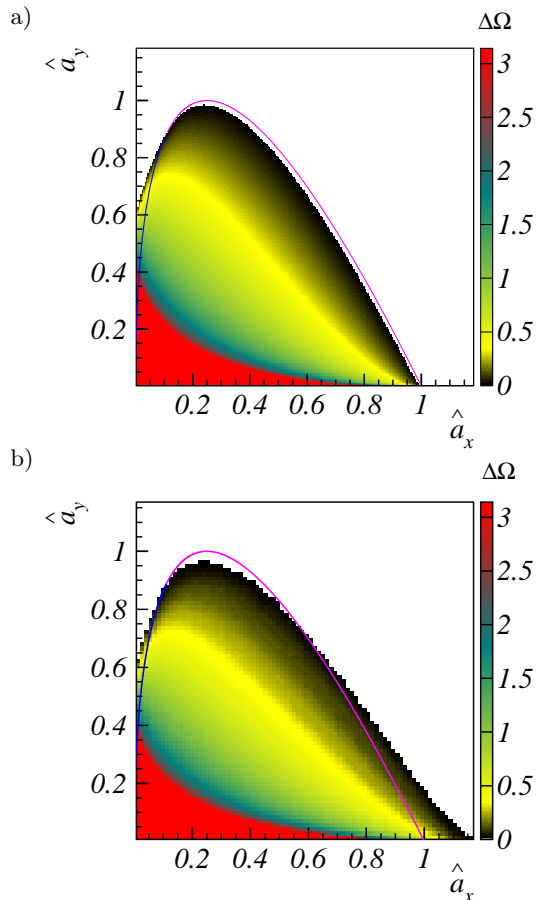
a)

b)

Figure 2: Complete stability domain. Part a): the stability domain as obtained by the analytic theory for the SIS18 lattice with a distribution of sextupolar errors. Part b) shows the stability domain of SIS18 obtained with simulations for one single sextupolar kick.

Part b) of Fig.(2) shows a comparison with computer simulation for the case with one single sextupolar error.

The widening of the stability domain shown in part b) of Fig.(2) is due to the effect of higher harmonics, which are all excited by a single localized error. Further simulations show that distributed errors generated to mainly excite a single harmonic component yields a better agreement between theory and simulations.

Fig.(2) shows another interesting feature: particles close to the $\hat{a}_y$ axis can be stable even "outside" the border of the stable 2D fix-lines. This demonstrate the existence of stable 2D "tori" as an analogon to the stable "islands" in 1D dynamics. Contrary to the 1D case these stable tori exist even without stabilizing detuning terms from higher order multipolar components.

In conclusion, in proximity to the coupled 2D resonance $Q_x + 2Q_y = N$ there are 2D fix-lines that are equivalent to fix-points in 1D. These lines are one dimensional closed curves in the 4D phase space. Their number is infinite, and they may either be stable or unstable. Our analysis is focused on distributions of sextupolar errors around a circular machine and should be seen as a general extension of previous studies [16, 17]. Our study links the stability domain to the existence of fix-lines. We also find that the properties of stability define a scaling law, naturally associated to the scaling coefficient $[4L\Lambda/(2\pi\Delta_r)]^2$. The general properties are summarized in Fig.(2) expressed in scaled emittances. The full derivation of this theory will be part of a future publication [13].

This study constitutes a milestone for the studies with space charge and the process of 2D fix-line crossing and trapping phenomena in 2D induced by space charge and in conjunction with synchrotron oscillation.

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